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1975 J. Phys. A: Math. Gen. 8 2000

(http://iopscience.iop.org/0305-4470/8/12/016)

J. Phys. A: Math. Gen., Vol. 8, No. 12, 1975. Printed in Great Britain.

Critical behaviour and cubic anisotropy

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Received 7 March 1975, in final form 16 July 1975

Abstract. The influence of the cubic anisotropy both in the quadratic and quartic part of an n-component 'spin' Hamiltonian is examined in the framework of a parquet-graph summation. For values of the anisotropy parameter f larger than a critical value, f_-, a first-order transition occurs which corresponds to the appearance of complex fixed points in the renormalization group approach. On the other hand for $f < f_-$ a first- or second-order transition occurs depending on f, n and the values of the bare coupling constants. For second-order transitions a quantity $m(n,f)$ plays the role of an effective number of components, $m(n, 0) = n$. For $m > 4$ and $m < -8$ the system has a strongly cubic behaviour whereas for $0 < m < 4$ a weakly cubic behaviour is observed; the latter changes to pure isotropic behaviour for $f \to 0$. For $-8 < m < 0$ the behaviour is strongly or weakly cubic depending on the values of the bare coupling constants. The (effective) critical exponents γ, α calculated to $O(\epsilon)$ or $O(n^0)$ depend on the anisotropy parameter f. The results give an explanation for the first-order transition in $KMnF_3$ and the decrease of the exponent β as T_c is approached in $SrTiO_3$.

1. Introduction

Many attempts have been made in recent years to elucidate the properties of anisotropic systems close to their critical points. In particular, the effect of cubic anisotropy has been considered within the renormalization group approach by several authors (eg Cowley and Bruce 1973, Wallace 1973, Ketley and Wallace 1973, Aharony 1973a,b,c, Brézin *et al* 1974). In fact, the anisotropy may change the critical behaviour of the system drastically. For instance, it may lead to a first-order transition (Wallace 1973) or to a different critical behaviour (Aharony 1973b). However, most of the papers are restricted to some cases of cubic anisotropy in the quartic (single-ion) term of the Hamiltonian breaking the full rotational symmetry. The reason is that one believes that the parameter f measuring the quadratic anisotropy becomes an irrelevant variable in the immediate vicinity of the critical point (Bruce 1974), although this result was obtained with some additional assumptions. But even if this is the case one expects that the anisotropy parameter f decays so slowly that there is a large region where the quadratic anisotropy has to be taken into account. Therefore, in this paper both the quadratic and the quartic (cubic) anisotropy will be considered above the transition temperature T_c in the framework of a parquet-graph expansion.

In the scaling region the results of this approach go over into the renormalization group results to order ϵ or n^0, respectively. Moreover, the parquet approximation leads to a description of the systems which is applicable in the whole critical domain up to the mean field region (Nattermann and Trimper 1974, Nattermann 1975). It is important

to note that our approach exploits none of the arguments of the different practical renormalization group calculations, eg there is no claim that the expressions exponentiate as in the ϵ-expansion. In fact, the small quantities of our perturbation theory are the vertex parts Γ_1 and Γ_2 proportional to the bare coupling constants.

The main advantage of our approach is that it also works in cases where the asymptotic critical region is not reached and a first-order transition occurs. This situation corresponds to the occurrence of complex fixed points or the absence of fixed points generally in the renormalization group scheme.

In order to get a larger variety of the behaviour of the system, here the number n of order-parameter components is assumed to be different from the space dimension d because of the introduction of an additional anisotropy. For the sake of completeness the case $n < 0$ is also considered by an analytic continuation with respect to n.

The paper is organized as follows: in § 2 the Hamiltonian for a system with cubic anisotropy is written in the notation of Aharony and Fisher (1973) in the reduced form. The Bethe–Salpeter equations for the renormalized coupling constants and Ward's identity for the inverse susceptibility are derived in the parquet approximation in § 3. To solve these integral equations it turns out that it is necessary to distinguish between the cases of weak and strong quadratic anisotropy. Moreover, the possibility of the occurrence of a first-order transition will be discussed in § 4. Section 5 is devoted to the calculation of the critical exponents. Further, the connection of our method to the fixed point equation of the renormalization group technique is established. Some applications of our results are discussed in § 6.

2. The Hamiltonian

The Hamiltonian for the description of an anisotropic cubic system may be written in the reduced form (Aharony and Fisher 1973)

$$\frac{H}{k_B T} = \frac{1}{2}\sum_{q,\alpha}(r_{0\alpha} + q^2 - f q_\alpha^2) Q_q^\alpha Q_{-q}^\alpha + \frac{1}{V}\sum_{q_1,q_2,q_3,\alpha,\beta}(u_0 + v_0\delta_{\alpha\beta}) Q_{q_1}^\alpha Q_{q_2}^\alpha Q_{q_3}^\beta Q_{-q_1-q_2-q_3}^\beta. \tag{2.1}$$

The Hamiltonian (2.1) is appropriate for the description of magnetic, ferroelectric or structural phase transitions. Corresponding to these cases the $\bar{n}$-component vector $\{Q_q^\alpha\}$ is interpreted as the classical spin vector, the normal coordinate of the soft mode or a staggered rotation angle, respectively.

Whereas one obtains the Hamiltonian (2.1) for the investigation of a magnetic system by adding a spin weighting function which restricts the fluctuations in the spin length, the Hamiltonian (2.1) emerges naturally from anharmonic lattice theory in the case of displacive type transitions (Cowley and Bruce 1973). In the latter case, as distinct from the magnetic case, the parameters u_0 and v_0 reflect the strength of the anharmonic potential and can be measured by various methods. The occurrence of anisotropy is manifested in two ways: through the anisotropy parameter f in the dispersion relation and through the quartic term with the prefactor v_0. Contrary to the previous papers on this subject here all values for f, u_0 and v_0 compatible with the stability requirements will be considered. For $1 > f \geqslant 0$ the correlations between different spins are 'pancake'-like whereas for $f < 0$ the correlations become needle shaped.

Because of the quadratic anisotropy in (2.1) which couples 'spin' and space variables one has to choose $\bar{n} = d$, where d denotes the space dimensionality. However, in accord

with the arguments of universality it would be desirable to relax the connection between $\bar{n}$ and d. This can be done—as proposed by Aharony (1973a)—by an appropriate choice of the parameter $r_{0\alpha}$. Namely, $r_{0\alpha} = r_0$ for $\alpha = 1, \ldots, n \leqslant d$ and $r_{0\alpha} = r_0 + \delta$ for the remaining $d-n$ components. As usual $\delta > 0$ and hence the latter (subdominant) components do not influence the critical behaviour. The advantage of disconnecting $\bar{n}$ from d is that, in the case of vanishing quadratic anisotropy, the results can be related directly to those obtained previously for quartic anisotropy only.

3. Perturbation theory

Restricting ourselves as usual to the static part of the propagator we define it by

$$G_{\alpha\beta}(q) = \langle Q_q^\alpha Q_{-q}^\beta \rangle = g(q)\delta_{\alpha\beta} \qquad 1 \leqslant \alpha, \beta \leqslant n \tag{3.1}$$

with

$$g_\alpha(q) = (r + q^2 - f q_\alpha^2)^{-1}. \tag{3.2}$$

Here, because of the self-consistency of our calculations r_0 is replaced by $r(r_0)$ with $r(r_{0c}) = 0$. To our degree of accuracy we do not obtain a renormalization of the q dependence of the propagator. (In the scaling region this renormalization is of order $O(\epsilon^2)$ and $O(n^{-1})$ whereas we restrict ourselves to $O(\epsilon)$ and $O(n^0)$, respectively.) In the usual perturbation theory the inverse susceptibility r is calculated by summing up certain classes of diagrams which are classified with respect to the unrenormalized vertex parts. However, it is well known that this procedure fails in the immediate vicinity of the critical point because near T_c the assumed smallness of the unrenormalized vertex part is cancelled by the smallness of r. This is the reason that we begin with the calculation of the renormalized vertex parts Γ_1, Γ_2 corresponding to the bare coupling constants u_0 and v_0, respectively. The first contribution to Γ_1, Γ_2 follows from the elementary polarization diagram which is evaluated in appendix 1. Within the parquet approximation all higher-order graphs are extracted from this elementary bubble (figure 1(*a*)) denoted by $A_{\alpha\beta}(r, p)$:

$$A_{\alpha\beta}(r, p) = \int \frac{d^d q}{(2\pi)^d} g_\alpha(q) g_\beta(q + p)$$

$$= (2\pi)^{-d} \int_0^\Lambda \int_\Omega \frac{q^{d-1}\, dq\, d\Omega_d}{[r + (q+p)^2 - f(q_\beta + p_\beta)^2](r + q^2 - f q_\alpha^2)}. \tag{3.3}$$

Here p denotes the external momentum. Depending on the possible ratios of the parameters r, f and the cut-off Λ, expression (3.3) reveals different behaviour. Indeed, for zero external momentum there are four different cases:

(*a*) $f > 0$ and $(1-f)\Lambda^2 \lesssim r \ll \Lambda^2$;
(*b*) $f < 0$ and $\Lambda^2 \lesssim r \ll (1-f)\Lambda^2$;
(*c*) $r \ll (1-f)\Lambda^2, \Lambda^2$;
(*d*) $\Lambda^2, (1-f)\Lambda^2 \ll r$.

In case (*a*) the behaviour is essentially $(d-1)$ dimensional whereas in case (*b*) one-dimensional behaviour is simulated. For temperatures close to T_c, r becomes sufficiently small and hence in case (*c*) a d-dimensional behaviour occurs. Case (*d*) is without interest because it corresponds to the region where the Landau theory is still valid. In

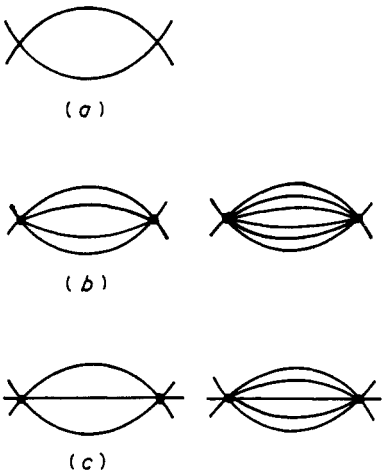

Figure 1. Lowest-order diagrams to the renormalized quartic (a, b) and higher-order coupling constants (c) arising from quartic (a) and higher-order interactions (b, c).

the following we restrict ourselves to region (c), eg to the vicinity of the critical point. Here we assume that regions (a) or (b) are small or are located entirely in the range where the Landau theory is applicable. We are forced to this since our perturbation theory works only in the region $2 < d \leqslant 4$. Moreover, if we exclude the borderline case $d = 4$ we can extend the range of integration to infinity ($\Lambda \to \infty$). From this fact we obtain a direct connection between the critical domain (c) and the mean field region.

Under these assumptions $A_{\alpha\beta}(r, p = 0)$ is calculated in appendix 1. The final result is

$$A_{\alpha\beta}(r, p = 0) = I_{\alpha\beta}(f) K_d x \tag{3.4}$$

where $x = \Gamma(2-\epsilon/2)\Gamma(\epsilon/2) r^{-\epsilon/2}$, $\epsilon = 4-d$, $I_{\alpha\beta}(f)$ is defined in (A.2):

$$I_{\alpha\beta}(f) = \rho(f)(1+\tfrac{1}{4}s(f)(\delta_{\alpha\beta}-1)); \qquad 0 \leqslant s(f) \leqslant 4. \tag{3.5}$$

The functions $\rho(f)$ and $s(f)$ are measures of the quadratic anisotropy and are presented in appendix 1, and in figure 2. To our degree of accuracy the renormalized coupling constants follow from the summation of all parquet diagrams (Larkin and Khmel'nitskii 1969). Extending this method to the summation of non-logarithmic corrections (Ginzburg 1974, Nattermann 1975, Nattermann and Trimper 1974) one

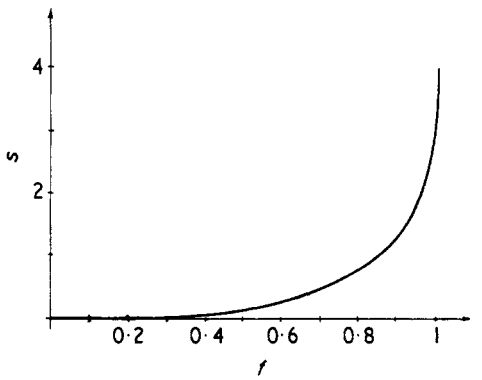

Figure 2. Anisotropy function $s(f)$ against f for $0 \leqslant f \leqslant 1$.

obtains for the renormalized vertex parts

$$\Gamma_{\alpha\beta\gamma\delta}(p^2=0)=\gamma_{\alpha\beta\gamma\delta}-12\frac{u_0}{u}\int\frac{\mathrm{d}^dq}{(2\pi)^d}[\Gamma_{\alpha\beta\mu\nu}(q^2)\Gamma_{\rho\lambda\gamma\delta}(q^2)+\Gamma_{\alpha\gamma\mu\nu}(q^2)\Gamma_{\rho\lambda\beta\delta}(q^2)$$
$$+\Gamma_{\alpha\delta\mu\nu}(q^2)\Gamma_{\rho\lambda\beta\gamma}(q^2)]G_{\mu\rho}(q)G_{\nu\lambda}(q) \tag{3.6a}$$

with

$$\gamma_{\alpha\beta\gamma\delta}=\tfrac{1}{3}u(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\alpha\gamma}\delta_{\beta\delta})+vg_{\alpha\beta\gamma\delta} \tag{3.6b}$$

$$\frac{u}{u_0}=\frac{v}{v_0}=12K_d\rho(f),\qquad g_{\alpha\beta\gamma\delta}=\begin{cases}1 & \alpha=\beta=\gamma=\delta\\ 0 & \text{otherwise.}\end{cases} \tag{3.6c}$$

The tensor structure of $\Gamma_{\alpha\beta\gamma\delta}$ is the same as (3.6*b*) with u and v replaced by Γ_1 and Γ_2, respectively. For zero external momentum the renormalized vertex parts depend only on r, ie on x. Then after some tedious calculations we get from (3.6*a*)

$$\Gamma_1(x)=u-\int_0^x\left(\frac{n+8-s}{3}\Gamma_1^2(y)+2\Gamma_1(y)\Gamma_2(y)\right)\mathrm{d}y \tag{3.7a}$$

$$\Gamma_2(x)=v-\int_0^x(3\Gamma_2^2(y)+4\Gamma_1(y)\Gamma_2(y)+\tfrac{1}{3}s\Gamma_1^2(y))\,\mathrm{d}y \tag{3.7b}$$

with $0\leqslant s<4$. The equations (3.7) include some cases considered previously by several authors. For vanishing cubic anisotropy Γ_2 becomes zero and Γ_1 reduces to the renormalized coupling constant of the isotropic n-vector model (Wilson and Kogut 1974). In the opposite case of strong anisotropy $s\to 4$ and for $n=3$ equations (3.7) change over into those considered recently by Khmel'nitskii and Shneerson (1973) for a three-dimensional system at low temperatures. One sees further from equation (3.7*b*) that the anisotropic form of the dispersion relation creates an anisotropy of the vertex part $\Gamma_{\alpha\beta\gamma\delta}$ even in the case of vanishing v. This fact was already observed by Aharony and Fisher (1973). On the other hand the vertex part anisotropy does not influence the dispersion relation (to our degree of accuracy). This is the main reason that we consider both kinds of anisotropy.

From physical grounds stated in the previous section it follows that $s\neq 0$ is meaningful only for $1<n\leqslant 4$. However below we will treat equations (3.7) formally with $s\neq 0$ for all n and take—if necessary—$s=0$ only in the final expression. The continuation to $n<0$ was considered in detail by Fisher (1973). For $n=1$, Γ_1 and Γ_2 are no longer independent coupling constants and in the equation for the full vertex part $\tilde{\Gamma}=\Gamma_1+\Gamma_2$, s does not appear (see appendix 3).

For our further calculations it is important to note that $\Gamma_1(x)$ and u have the same signs for all x. Indeed, in the case where $\Gamma_1(x_1)=0$, all derivatives of $\Gamma_1(x)$ also vanish at x_1 (because of equation (3.7)) leading to $\Gamma_1(x)\equiv 0$ for all values x, in contradiction with $\Gamma_1(0)=u\neq 0$. For the sake of completeness we remark that for $u=\Gamma_1=0$ the system decays into n Ising systems, each of which corresponds to $n=1$.

The stability of the system is related to the boundedness from below of the free energy which requires the positivity of the interaction part:

$$\Gamma_{\alpha\beta\gamma\delta}Q_0^\alpha Q_0^\beta Q_0^\gamma Q_0^\delta>0. \tag{3.8}$$

Writing $Q_0^\alpha=Q_0n_\alpha$ with $\Sigma_{\alpha=1}^n n_\alpha^2=1$ we get from (3.8)

$$\Gamma_1+\Gamma_2>0 \tag{3.9a}$$

$$\Gamma_1+(1/n)\Gamma_2>0 \tag{3.9b}$$

where the inequalities (3.9) become essential for $u > 0$ and $u < 0$, respectively. If one of these inequalities is violated interaction terms proportional to $w_n(Q^2)^n$ higher than fourth order have to be taken into account. Here, we will assume that all the contributions from such Q^{2n} terms ($n > 2$), giving rise to a renormalized transition temperature and effective fourth-order coupling constants, are already included in r_{0c}, u and v, respectively. (Below we suppose that the stability conditions are always fulfilled at $x = 0$, ie $u+v > 0$, $u+v/n > 0$.) The perturbation theory for the Γ_i incorporating the remaining contributions from the Q^{2n} terms will again be developed from the second-order graphs. These graphs include $2(n-1)$ internal lines connecting two w_n vertices (figure 1(*b*)). A simple power counting shows then, that these graphs become less important than the bubble (figure 1(*a*)). Indeed, we obtain for the leading contributions to the Γ_i: constant $\times w_n^2 r^{2(n-2)-\epsilon(n-3/2)}$, $n > 2$. It follows that the inclusion of higher-order interaction terms cannot change the sign of the expression (3.8). However, in the case where (3.8) becomes negative, owing to the quartic interaction, before the transition point is reached, these additional terms lead to a first-order transition (such a conclusion was also drawn by Khmel'nitskii and Shneerson 1973). A corresponding effect in the Bethe–Salpeter equations for the renormalized $(Q^2)^n$ interaction does not occur in the case where $\epsilon < 1$ because their leading contributions are terms proportional to $w_n^2 r^{\frac{1}{2}d(n-1)-n}$ (figure 1(*c*)). They become important only for $d \leqslant 2n/(n-1)$ in the case of polycritical phenomena (Chang *et al* 1974).

The Green functions can be obtained from Γ_1 and Γ_2 from Ward's identity (Larkin and Khmel'nitskii 1969):

$$\partial G_{\alpha\beta}^{-1}(r)/\partial r_0 = \Lambda_{\alpha\beta}(r) \tag{3.10}$$

$$\Lambda_{\alpha\beta}(r) = \delta_{\alpha\beta} - 12(u/u_0)(2\pi)^{-d} \int_r^\infty \mathrm{d}^d q \Gamma_{\alpha\beta\gamma\delta}(q^2) G_{\gamma\mu}(q) G_{\delta\lambda}(q) \Lambda_{\mu\lambda}(q). \tag{3.11}$$

With $\Lambda_{\alpha\beta}(r) = \Lambda(x)\delta_{\alpha\beta}$ and equations (3.1)–(3.6*c*) we get after differentiation with respect to x

$$\frac{\partial \Lambda(x)}{\partial x} = -[\tfrac{1}{3}(n+2)\Gamma_1(x) + \Gamma_2(x)]\Lambda(x) \tag{3.12}$$

or

$$\ln \Lambda(x) = -\int_0^x [\tfrac{1}{3}(n+2)\Gamma_1(y) + \Gamma_2(y)]\, \mathrm{d}y. \tag{3.13}$$

We see that the dispersion anisotropy enters equation (3.13) only through equations (3.7). Both equations, (3.7) and (3.13), describe the behaviour of the system in the whole transition region.

4. Discussion of the integral equations

In order to solve the system of integral equations (3.7) it turns out to be convenient to introduce the quantities

$$c = \tfrac{1}{6}(n-4-s) \tag{4.1}$$

$$b^2 = \tfrac{1}{3}s - c^2 = \tfrac{1}{36}(s_+ - s)(s - s_-) \tag{4.2}$$

$$s_\pm = n + 2[1 \pm \sqrt{3}\sqrt{(n-1)}]. \tag{4.3}$$

The values of f corresponding to $s_\pm$ will be denoted by $f_\pm$. From (4.2) it is natural to distinguish the following three main cases which exhibit different critical behaviour as we will see below:

$$\text{(i) } b^2 > 0, \qquad \text{(ii) } b^2 < 0, \qquad \text{(iii) } b^2 = 0. \tag{4.4}$$

The different domains of the n–$s(f)$ plane corresponding to these cases are depicted in figure 3.

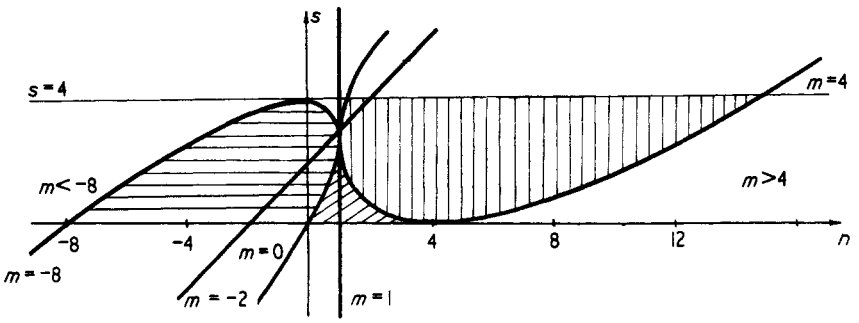

Figure 3. Plane of the anisotropy function s and the number of components n with the curves $m(n, s)$ = constant. The domain of case (i) is depicted by the vertical hatching. The horizontal hatching encloses the domain where strong or weak cubic behaviour is found depending on the initial conditions ($-8 < m < 0$). The slanting hatching represents the weakly cubic region ($0 < m < 4$). Case (iii) coincides with the curve $m = 4$ and case (ii) corresponds to the remaining areas.

We start with the discussion of case (i) which also includes the case of strong anisotropy for $2 \leqslant n \leqslant 4$ by defining a new function $\psi(x)$:

$$\Gamma_2 = (\psi + c)\Gamma_1. \tag{4.5}$$

Differentiating equations (3.7) and (4.5) with respect to x and using

$$\Gamma_2'/\Gamma_1' = \psi + c + \Gamma_1\psi'/\Gamma_1'$$

we find after some calculations

$$\Gamma_1 = u\frac{\psi^2+b^2}{\gamma^2+b^2}\exp\left\{\frac{4(c+1)}{b}\left[\tan^{-1}\left(\frac{\psi}{b}\right)-\tan^{-1}\left(\frac{\gamma}{b}\right)\right]\right\} \tag{4.6}$$

with

$$\gamma = \psi(0) = \frac{v}{u} - c.$$

From equations (3.7), (4.5) and (4.6) one gets further

$$\psi'(x) = f(\psi) = -u\frac{(\psi^2+b^2)^2}{\gamma^2+b^2}\exp\left\{\frac{4(c+1)}{b}\left[\tan^{-1}\left(\frac{\psi}{b}\right)-\tan^{-1}\left(\frac{\gamma}{b}\right)\right]\right\}. \tag{4.7}$$

Equation (4.7) always leads to unstable solutions. Indeed, considering to begin with $u > 0$, $f(\psi)$ is negative for all ψ and hence $\psi(x)$ becomes $-\infty$ at

$$x_0 = \int_\gamma^{-\infty} \frac{d\psi}{f(\psi)}. \tag{4.8a}$$

For $x \to x_0$ we find from equations (4.5)–(4.8*a*)

$$\psi(x) \propto -(x_0-x)^{-1/3}, \qquad \Gamma_1(x) \propto (x_0-x)^{-2/3}, \qquad \Gamma_2(x) \propto -(x_0-x)^{-1},$$

ie the renormalized vertex parts diverge for $x \to x_0$. As a consequence, our starting equations (3.7) are no longer valid since for large Γ_i one cannot restrict oneself to the lowest-order contributions in the Bethe–Salpeter equations (3.7).

However, before the point x_0 is reached the system becomes unstable at x^* due to the violation of the stability condition (3.9*a*):

$$x^* = \int_\gamma^{-1-c} \frac{\mathrm{d}\psi}{f(\psi)}. \tag{4.9a}$$

Similarly, for $u < 0$, $\psi(x)$ becomes infinite at

$$x_0 = \int_\gamma^{\infty} \frac{\mathrm{d}\psi}{f(\psi)} \tag{4.8b}$$

hence $\psi(x) \propto (x_0-x)^{-1/3}$, $\Gamma_1(x) \propto (x_0-x)^{-2/3}$, $\Gamma_2(x) \propto (x_0-x)^{-1}$ for $x \to x_0$. Here the inequality (3.9*b*) is violated at

$$x^* = \int_\gamma^{-n-c} \frac{\mathrm{d}\psi}{f(\psi)}. \tag{4.9b}$$

Thus, a sufficiently large quadratic anisotropy in the range $1 < n < 15$ (figure 2) leads to an instability before the critical point is reached. An estimation shows that $\Gamma_1(x^*)$, $\Gamma_2(x^*)$ are of order u, hence our basic equations (3.7) are valid up to the instability point. It is easy to see that x^* increases with decreasing b and becomes ∞ for $b = 0$, ie the transition changes from first to second order. Therefore for small b there is a large temperature region not too close to the transition point where the system seems to undergo a second-order transition. Actually, in the case where $\gamma u > 0$, $\psi(x)$ will change its sign and then for $\psi(x) \lesssim 0$ there is a domain where the integration of equation (4.7) yields

$$4(c+1)xu\frac{\psi^2+b^2}{\gamma^2+b^2}\exp\left\{\frac{4(c+1)}{b}\left[\tan^{-1}\left(\frac{\psi}{b}\right)-\tan^{-1}\left(\frac{\gamma}{b}\right)\right]\right\} = 1. \tag{4.10}$$

From this solution we are able to calculate critical exponents as will be shown in § 5.

As it turns out, case (ii) becomes more complex. Because of this it is convenient to change over to new vertex parts Γ and Γ_A by means of the transformation

$$\Gamma = (1+\lambda)\Gamma_1 \qquad \Gamma_A = \Gamma_2 - \lambda\Gamma_1 \tag{4.11}$$

with

$$\lambda = \lambda(n,s) = c \pm \sqrt{(c^2 - \tfrac{1}{3}s)} \equiv c \pm (|c| - \lambda_0). \tag{4.12}$$

Inserting (4.11) into (3.7) one obtains a new set of equations for an m-component system including only vertex part anisotropy, but with a non-integer number m of components:

$$\begin{aligned} \Gamma(x) &= u_\lambda - \int_0^x [\tfrac{1}{3}(m+8)\Gamma^2(x) + 2\Gamma(x)\Gamma_A(x)] \\ \Gamma_A(x) &= v_\lambda - \int_0^x (3\Gamma_A^2(x) + 4\Gamma(x)\Gamma_A(x)) \\ m &= m(n,s,\lambda) = \frac{n-s-2\lambda}{1+\lambda} \qquad u_\lambda = (1+\lambda)u \qquad v_\lambda = v - \lambda u. \end{aligned} \tag{4.13}$$

The two solutions of equation (4.12) are connected with each other by the transformation

$$\lambda_1 \to \lambda_2 = -\lambda_1 + 2c \tag{4.14a}$$

which yields

$$m(\lambda_1) \to m(\lambda_2) = \frac{m(\lambda_1)+8}{m(\lambda_1)-1} \tag{4.14b}$$

The curves of $m(n, s) =$ constant are the parabolas

$$s = (n+2) \pm (m+2)\left(\frac{n-1}{m-1}\right)^{1/2} \qquad m \neq 1 \tag{4.15}$$

with vertices located on the line $s+3n-6=0$. Because of (4.14) $m(\lambda_1)$ and $m(\lambda_2)$ belong to different sections of the same parabola (figure 3). Note that the transformation (4.14) translates the regions $m < -8$, $-8 < m < -2$ and $m > 4$ to $0 < m < 1$, $-2 < m < 0$ and $1 < m < 4$, respectively, and vice versa ($m = -2, 4$ remain unchanged under this transformation). For our further calculations it is convenient to choose $\lambda_1 = \lambda_0 c/|c|$ from (4.12) since in this case $\lambda_1 = 0$ for $s = f = 0$, ie $m(n, s=0) = n$.

Starting from the *ansatz*

$$\Gamma_A(x) = \tfrac{1}{6}(m-4)(\phi(x)+1)\Gamma(x) \qquad m \neq 4 \tag{4.16}$$

we get, after calculations similar to those in case (i),

$$\phi'(x) = \phi'(0)\left(\frac{\phi(x)-1}{\phi(0)-1}\right)^{2(1+\theta)}\left(\frac{\phi(x)+1}{\phi(0)+1}\right)^{2(1-\theta)} \tag{4.17}$$

and

$$\Gamma(x) = u_\lambda\left(\frac{\phi(x)-1}{\phi(0)-1}\right)^{1+2\theta}\left(\frac{\phi(x)+1}{\phi(0)+1}\right)^{1-2\theta} \tag{4.18}$$

with

$$\begin{aligned} &\phi(0) = \rho \qquad \phi'(0) = -v_\lambda(\rho-1) \\ &\rho+1 = \frac{6v_\lambda}{(m-4)u_\lambda} = (\theta-1)\frac{v_\lambda}{u_\lambda} \\ &\theta = \frac{n+2-s}{n-s-4-6\lambda} = \frac{m+2}{m-4}. \end{aligned} \tag{4.19}$$

We shall study these equations for arbitrary θ and any values $\phi(0)$, $\phi'(0)$ compatible with the stability requirements. Two points are worth noting:

(i) The equations (4.17), (4.18) are invariant with respect to the transformation θ, ϕ to $-\theta$, $-\phi$. Hence we can restrict ourselves to the case $\theta > 0$ in the following.

(ii) Under the transformation (4.14) θ, $\phi(0)$ and $\phi'(0)$ become $-\theta$, $-\phi(0)$ and $-\phi'(0)$, respectively, resulting in $\phi(x;\lambda_1) = -\phi(x;\lambda_2)$. On the other hand, as one would expect, the physical quantities Γ_1, Γ_2 remain unchanged under this transformation.

With this new notation the stability conditions read

$$\phi \gtrless -\theta \qquad \text{for } u_\lambda(\theta-1) \gtrless 0$$

and

$$\phi \gtrless -\theta\frac{c+n}{c+1} \qquad \text{for } u_\lambda n(\theta-1) \gtrless 0. \tag{4.20}$$

Next we consider the qualitative behaviour of the solutions for large x. For $\theta > 0$, $\phi'(x)$ has a zero at $\phi = 1$ and a pole or a zero at $\phi = -1$ depending on whether $\theta > 1$ or $1 > \theta > 0$. For these cases $\phi'(x)/\phi'(0)$ against $\phi(x)$ is schematically drawn in figure 4. From this we are able to estimate the behaviour of $\phi(x)$ for large x depending on the initial conditions $\phi(0) = \rho$ and $\phi'(0) = -v_\lambda(\rho-1)$. Because of the simplicity of the discussion we think readers will repeat it by themselves more quickly than by following a lengthy description. The result is depicted in figure 5 where we have used the more

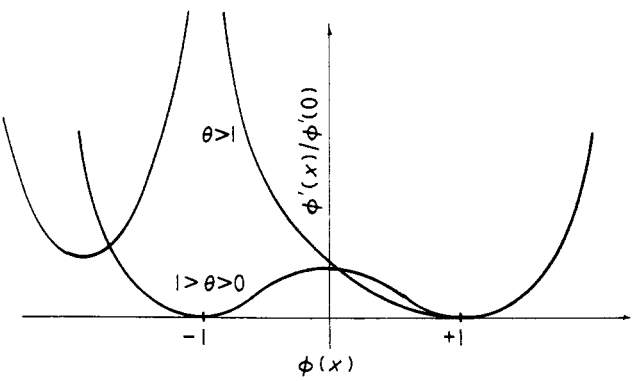

Figure 4. Schematic representation of the function $\phi'(x)/\phi'(0)$ against $\phi(x)$.

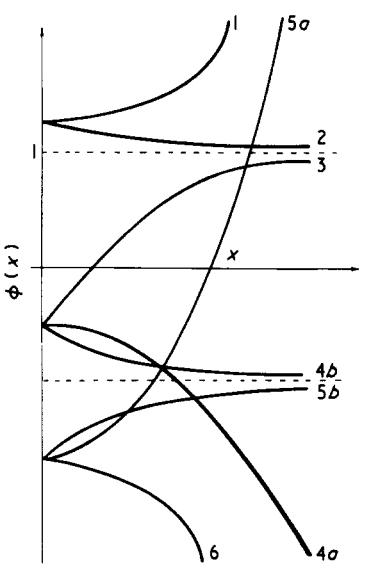

Figure 5. Schematic representation of the function $\phi(x)$ against x. The cases 1 to 6 are listed in table 1.

convenient classification $\theta > \frac{1}{2}, \frac{1}{2} > \theta > 0$. The different curves belong to the cases listed in table 1. The case $\theta < 0$ follows from the substitution of θ and ϕ by $-\theta$ and $-\phi$, respectively. From the initial conditions $\phi(0)$, $\phi'(0)$ and θ given there one obtains the corresponding values of the bare coupling constants by means of relations (4.13) and (4.19). Comparing the solutions listed in table 1 with the stability conditions (4.20) it turns out that only the solutions with ϕ finite are stable. According to what we have said in § 3 the remaining solutions hence lead to a first-order transition. They include also the case of large n and $u > 0$, $v < 0, f = 0$ considered recently by Wallace (1973). For large n we have $\theta > 1$ and $u > 0$, $v < 0$: $\phi(0) < -1$, $\phi'(0) < 0$, ie curve 6 applies, and leads in fact to a first-order transition.

Table 1.

Initial values	$\theta > \frac{1}{2}$ ($m > 4$ or $m < -8$)	$\frac{1}{2} > \theta > 0$ ($-8 < m < -2$)
$\phi(0) > 1$ $\begin{cases}\phi'(0) > 0\\ \phi'(0) < 0\end{cases}$	(1) $\phi \to \infty$ (2) $\phi \to 1$	(1) $\phi \to \infty$ (2) $\phi \to 1$
$1 > \phi(0) > -1$ $\begin{cases}\phi'(0) > 0\\ \phi'(0) < 0\end{cases}$	(3) $\phi \to 1$ (4*a*) $\phi \to -\infty$	(3) $\phi \to 1$ (4*b*) $\phi \to -1$
$\phi(0) < -1$ $\begin{cases}\phi'(0) > 0\\ \phi'(0) < 0\end{cases}$	(5*a*) $\phi \to +\infty$ (6) $\phi \to -\infty$	(5*b*) $\phi \to -1$ (6) $\phi \to -\infty$

We conclude this section with a remark on case (iii). That can be treated easily by taking the corresponding limits $b^2 \to 0$ and $m \to 4$ in the final expressions of cases (i) and (ii), respectively (see also the following section). From equations (4.6), (4.7) one gets

$$\Gamma_1 = u\frac{\psi^2}{\gamma^2}\exp\left[-4(c+1)\left(\frac{1}{\psi}-\frac{1}{\gamma}\right)\right]$$
$$\psi'(x) = -u\frac{\psi^4}{\gamma^2}\exp\left[-4(c+1)\left(\frac{1}{\psi}-\frac{1}{\gamma}\right)\right]. \tag{4.21}$$

For $u < 0$ the same arguments as in case (i) can be applied leading to a first-order transition. However, for $u > 0$, $\psi(x)$, $\Gamma_1(x)$ become zero for large x and a second-order transition occurs (see § 5).

5. Scaling region

We saw in the last section that only those solutions ϕ, and hence Γ_1, Γ_2, which remain finite for large x, are compatible with the stability requirements. By solving the differential equations (4.7) and (4.17), respectively, exactly in these cases we would get a description of the system in the whole critical domain from the mean field region up to the critical point (ie from $x = 0$ to $x = \infty$). This could be done only by numerical integration which we do not intend here (with the exception of the case $n = 1$ considered in appendix 3). Fortunately, in the large-x limit the behaviour of ϕ is dominated by only one power of x.

If we look for the stable solutions of case (ii) $\phi \to \pm 1$ we can substitute $\phi \pm 1$ by ± 2 in equation (4.17). Then this equation can be solved easily. First we treat the case $\phi \to 1$, this means $m > 4$ ($u > 0$, $v_\lambda > 0$) or $m < 0$ ($v_\lambda > 0$, $u < 0$), respectively. We get

$$\phi - 1 = (A + Bx)^{-1/(1+2\theta)} \tag{5.1}$$

with

$$A = (\rho - 1)^{-(1+2\theta)} \tag{5.2}$$

$$B = -(1+2\theta)\phi'(0)(\rho-1)^{-2(\theta+1)}(\rho+1)^{2(\theta-1)}2^{2(1-\theta)}$$

and hence

$$\Gamma(x) \simeq \frac{1}{mx}$$

$$\Gamma_1(x) \simeq \frac{1}{2(3c+2-\lambda_1)x} \qquad \Gamma_2(x) \simeq \frac{2c-\lambda_1}{2(3c+2-\lambda_1)x} \tag{5.3a}$$

ie the renormalization preserves the cubic anisotropy in the quartic term even in the case $\lambda_1 \to 0$. Therefore we will designate this behaviour as strongly cubic. Inserting this result in equation (3.13) one obtains

$$\Lambda(x) \simeq \text{constant} \times x^{-(n+2+6c-3\lambda_1)/3(n-s-2\lambda_1)}.$$

Taking into account that $r = (r_0 - r_{0c})\Lambda(x)$ and equation (3.4) it follows for the critical exponent γ that

$$\gamma^{-1} = 1 - \frac{2(n-1)-s-3\lambda_1}{6(n-s-2\lambda_1)}\epsilon \equiv \gamma_c^{-1}. \tag{5.4a}$$

The specific heat is proportional to the polarization operator (Larkin and Khmel'nitskii 1969):

$$C_v \propto \int \Lambda^2(x)\,dx \propto \frac{n-s-2\lambda_1}{4-n-s}x^{(4-n-s)/3(n-s-2\lambda_1)}$$

and hence the critical exponent α becomes

$$\alpha = \gamma_c \frac{4-s-n}{6(n-s-2\lambda_1)}\epsilon \equiv \alpha_c. \tag{5.5a}$$

Note that $n = 4-s$ corresponds to a logarithmic divergence of the specific heat. For $s = \lambda_1 = 0$ these are the exponents of the cubic fixed point of the renormalization group approach (Aharony 1973b). For small s one obtains from (5.4*a*)

$$\gamma_c^{-1} \simeq 1 - \frac{n-1}{3n}\left(1 - \frac{(n-1)(n+4)}{6n(n-4)}s\right)\epsilon \tag{5.6a}$$

where we have used $\lambda_1 \simeq s/(n-4)$. For $n > 4$ this corresponds to a decrease of the exponent γ due to the quadratic anisotropy compared with the pure quartic anisotropy values. However, the physical meaning of this result is unclear because one must take $n \leqslant d < 4$ when considering this anisotropy.

The case $\phi \to -1$ corresponds to the regions $0 < m < 4$ ($u > 0, v_\lambda > 0$ or $u_\lambda > [3/(m-4)]v_\lambda > 0$) and $-8 < m < 0$ ($u > 0, v_\lambda > 0$ or $u_\lambda > [3/(m-4)]v_\lambda > 0$) in

the n–s plane (figure 3). The results can be derived from equation (4.17) in the same way as for $\phi \to 1$ or more directly from the expressions (5.1)–(5.5*a*) by means of the transformation (4.14):

$$\Gamma_1(x) \simeq \frac{1}{2(c+2+\lambda_1)x} \qquad \Gamma_2(x) \simeq \frac{\lambda_1}{2(c+2+\lambda_1)x} \tag{5.3b}$$

$$\gamma^{-1} = 1 - \frac{n+2+3\lambda_1}{n+8-s+6\lambda_1}\frac{\epsilon}{2} \equiv \gamma_H^{-1} \tag{5.4b}$$

$$\alpha = \gamma_H \frac{4-s-n}{n+8-s+6\lambda_1}\frac{\epsilon}{2} \equiv \alpha_H. \tag{5.5b}$$

If $s = \lambda_1 = 0$, $\Gamma_2(x)$ becomes zero and the exponents are those of the Heisenberg fixed point (Wilson 1972). On the other hand, for $s, \lambda_1 \neq 0$ both Γ_1 *and* Γ_2 are different from zero and the behaviour is weakly cubic. For small s we have from (5.4*b*)

$$\gamma_H^{-1} \simeq 1 - \frac{n+2}{n+8}\left(1 + \frac{s(n-1)}{n+8}\right)\frac{\epsilon}{2}.$$

That means the anisotropy manifests itself in an increase of γ.

Case (iii) will be treated in the same manner. We integrate equation (4.21) and take into account only the leading contribution to $\psi(x)$ for large x. With $s = s_-$ one gets

$$\Gamma_1(x) \simeq \frac{\sqrt{3}}{4\sqrt{(n-1)}x} \qquad \Gamma_2(x) \simeq \frac{2\sqrt{(n-1)}-\sqrt{3}}{4\sqrt{(n-1)}x} \qquad n \neq 1. \tag{5.3c}$$

This result can also be obtained from the expressions (5.3*a*, *b*) by taking the limit $m \to 4$, ie setting $\lambda_1 = c$. Thus

$$\gamma_H^{-1}(m=4) = \gamma_c^{-1}(m=4) = 1 - \left[1 + \left(\frac{n-1}{3}\right)^{1/2}\right]\frac{\epsilon}{8} \tag{5.4c}$$

$$\alpha_H(m=4) = \alpha_c(m=4) = \gamma_H\left[1 - \left(\frac{n-1}{3}\right)^{1/2}\right]\frac{\epsilon}{4}. \tag{5.5c}$$

Case (i) does not include solutions of the type considered above, ie the asymptotic critical region will never be reached. Nevertheless, there is a region where scaling holds in a restricted sense. Indeed, one obtains from equations (4.5), (4.6) and (4.10) for $\gamma u > 0$

$$\Gamma_1(x) \simeq \frac{1}{4(c+1)x} \qquad \Gamma_2(x) \simeq \frac{c}{4(c+1)x} \tag{5.3d}$$

hence

$$\gamma^{-1} = 1 - \frac{3n-s}{n+2-s}\frac{\epsilon}{8} \tag{5.4d}$$

$$\alpha = \gamma \frac{4-s-n}{n+2-s}\frac{\epsilon}{4}. \tag{5.5d}$$

Note that case (i) ($b^2 > 0$) can likewise be treated as case (ii) ($b^2 < 0$) but with $\lambda = c \pm ib$ complex. In the case of negligibly small imaginary part ib in fact one obtains the results (5.3*d*)–(5.5*d*) from the corresponding formulae (5.3*a*)–(5.5*b*).

If one is only interested in the solutions for Γ_1, Γ_2 in the scaling region one can make use of the fact that Γ_1, Γ_2 vanish like $r^{\epsilon/2}$ for $r \to 0$ (Wilson and Kogut 1974). With the *ansatz* $\Gamma_i(x) = \epsilon^{-1} x^{-1} \gamma_i(x)$ one obtains the fixed points γ_i^* from the condition $\partial \gamma_i / \partial x = 0$ (appendix 2). The solutions $\Gamma_i(x) = \epsilon^{-1} x^{-1} \gamma_i^*$ agree with those stated in the equations (5.3*a–c*). In case (i) the fixed points become complex. Comparing this result with the discussion of case (i) in § 4 we conclude that the occurence of complex fixed points corresponds to a first-order transition. A further remark is concerned with the unstable solutions of (ii) and (iii). In these cases the fixed points would be real but, however, the scaling region is not reached because of the violation of the stability conditions (4.20). The values of the bare coupling constants which allow a second-order transition are depicted in figure 6 for the interesting case $m > 1$.

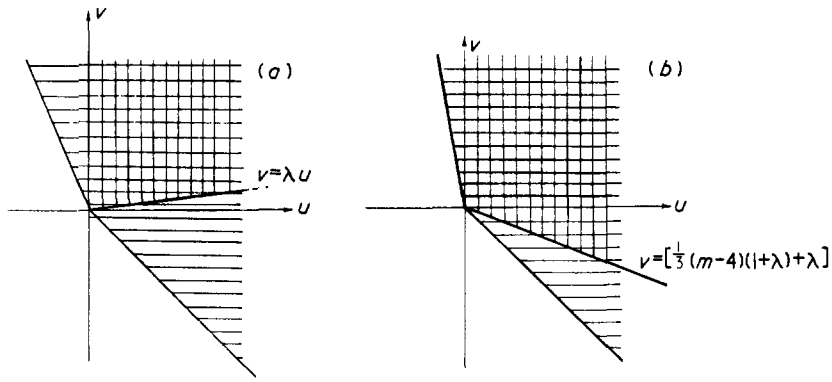

Figure 6. The plane of the bare coupling constants. The domain of the positivity of the bare interaction is depicted by the horizontal hatchings. The vertical hatching shows the domain where a second-order transition is permitted for (*a*) $m > 4$ and (*b*) $1 < m < 4$.

The expressions (5.4) and (5.5) are in agreement with the scaling law $d\nu = 2 - \alpha$ ($\eta \equiv 0$). The critical behaviour considered above differs in two ways from that one would expect from universality:

(i) A second-order transition occurs only for a definite set of initial values of the bare coupling constants and of the anisotropy parameter f (see figures 6 and 3, respectively). This was already observed by Brézin *et al* (1974) for the case $f = 0$.

(ii) The critical exponents depend on the non-universal anisotropy function $s(f)$ as well as the dimension d and the number of order-parameter components n. But this is not so strange since the insertion of an anisotropy in the quadratic term of the Hamiltonian lowers the symmetry of the interaction. A similar situation arises in systems with an additional dipolar interaction (Fisher and Aharony 1973, Nattermann and Trimper 1974). Since in a range very near to the transition point one has to regard higher-order contributions than considered here (ie $O(\epsilon^2)$, $O(n^{-1})$) which may lead to a decrease of f, one can interpret the critical exponents derived above as effective critical exponents as a precaution. Thus it is possible that the true asymptotic exponents are in fact independent of f. But they are just the effective critical exponents which are interesting from the experimental point of view.

6. Conclusion

The original aim of this paper was to estimate the influence of cubic anisotropy both in the quadratic and quartic part of the Hamiltonian on the critical behaviour. The main results were:

(i) For sufficiently large values of the anisotropy parameter f ($f > f_-$) and $n > 1$ one always obtains a first-order transition (case (i)). In the renormalization group approach this corresponds to the occurrence of complex fixed points. However, in certain circumstances one is able to calculate critical exponents for temperatures not too close to the transition point.

(ii) In case (ii) it is appropriate to introduce a quantity $m(n, s(f))$ which plays the role of an effective number of components. For $m > 4$ and $m < -8$ the system displays strongly cubic behaviour, whereas for $0 < m < 4$ the behaviour is weakly cubic passing over to Heisenberg-like behaviour for vanishing quadratic anisotropy.

(iii) For $-8 < m < 0$ one observes the interesting fact that the behaviour becomes strongly or weakly cubic depending on the values of the bare coupling constants.

Firstly we consider the case $2 \leqslant n < 4$ where the influence of the quadratic anisotropy becomes drastic since it leads to a first-order transition already for small values of $s(f)$. For $n = 3$ one gets $s_- = 0{\cdot}1$ which yields $f_- = 0{\cdot}42$ as the critical value for f. These values are clearly smaller than those known for some substances undergoing a structural phase transition. However, for temperatures very close to the transition point one also has to take account of higher-order terms in the perturbation series (ie order ϵ^2) which lead to a renormalization of f. In the case where f becomes smaller in this way there will be a competition between the occurrence of a first-order transition and a change-over from case (i) to (ii). It appears likely that the smaller the initial value of f the more this mechanism will work. This leads to the possibility of explaining the first-order transition in $KMnF_3$ and the second-order one in $SrTiO_3$ although in both cases the initial values for f lie above the critical value f_-. Indeed, one has $f \simeq 1$ (Gesi *et al* 1972) and $f \simeq 0{\cdot}96$ (Stirling 1972) for $KMnF_3$ and $SrTiO_3$, respectively. (Remember that a small change in f corresponds to a large one in $s(f)$ for $f \lesssim 1$ (figure 2).) A detailed calculation on this mechanism including the renormalization of the anisotropy parameter f will be given in a forthcoming paper.

In the case of a second-order transition the exponents γ, α depend weakly on the anisotropy parameter f. However, if one includes the case of a first-order transition the influence of f becomes more pronounced. Then for $SrTiO_3$ one has, in the scaling region of case (i) with $n = 3$ and $f \simeq 0{\cdot}96$ (ie $s = 2{\cdot}13$), $\gamma \simeq 1{\cdot}43$. Using the scaling relation $\beta = (d-2)\gamma/4$ ($\eta \equiv 0$) this leads to $\beta = 0{\cdot}36$ compared to $\gamma = 1{\cdot}29$ and $\beta = 0{\cdot}32$ for $s = f = 0$. Such a decrease of the critical exponent was actually observed in $SrTiO_3$ by K A Müller, who reported it in a talk given at Leipzig University in May 1974.

For $n = 0$ the isotropic n-vector model allows a description of the self-avoiding walk problem (de Gennes 1972, des Cloizeaux 1974). It is not clear to the authors whether the anisotropic n-vector model considered above possesses any relevance for a modified version of this problem. As can be seen from figure 3 the case $n = 0$ exhibits a special variety of the behaviour.

After this work had been performed we were informed of a recent paper of Liukziutov and Prokrovskii (1975) who treat the special case $f = 0$, $n = 2$ of our model. With reference to the values of the bare coupling constants which lead to a first- or second-order transition, respectively, these authors came to the same conclusions as in the present paper. This is remarkable since Brézin *et al* (1974) found a smaller domain

for the bare coupling constants permitting a second-order transition in the case $n < 4$ (in particular $v_0 > 0$) than Liukziutov and Prokrovskii (1975) for $n = 2$ and the present authors for all $0 < n < 4$ (see figure 6).

The conclusion was drawn by Liukziutov and Prokrovskii (1975) and other authors (eg Aharony 1974) that the various types of changeover from second- to first-order transitions by means of the variation of the bare coupling constants correspond to the occurrence of tricritical points. Additionally we obtain our case (iii) as a new kind of tricritical point since we also consider the case $f \neq 0$. The corresponding tricritical exponents are thence given in equations (5.3*c*)–(5.5*c*). However, we believe that such kinds of tricritical points are less interesting than the usual Riedel–Wegner type because all phenomena arise as boundary cases of ordinary critical behaviour.

Acknowledgments

The authors thank Professor A Kühnel for many helpful discussions and a critical reading of the manuscript. They thank also Dr S Wendt for helpful advice and Dr A Aharony for sending a reprint of his talk at the Athens conference.

Appendix 1

For the evaluation of expression (3.3) we use the Feynman trick

$$(AB)^{-1} = \int_0^1 [(1-\alpha)A + \alpha B]^{-2}\, \mathrm{d}\alpha.$$

Then in the case of zero external momentum one gets

$$\begin{aligned} A_{\alpha\beta}(r, p=0) &= (2\pi)^{-d} \int_0^\infty \frac{q^{d-1}\,\mathrm{d}q}{(r+q^2)^2} \int_0^1 \mathrm{d}\alpha \int_\Omega \mathrm{d}\Omega_d [1 - f n_\alpha^2 - \alpha f (n_\beta^2 - n_\alpha^2)]^{-d/2} \\ &= K_d \Gamma(2-\epsilon/2)\Gamma(\epsilon/2) r^{-\epsilon/2} I_{\alpha\beta}(f); \\ n_\alpha &= \frac{q_\alpha}{|q|}. \end{aligned} \tag{A.1}$$

$I_{\alpha\beta}(f)$ includes the angular part of (A.1):

$$\begin{aligned} I_{\alpha\beta}(f) &= \tfrac{1}{2} K_d^{-1} (2\pi)^{-d} \int \mathrm{d}\alpha\, \mathrm{d}\Omega_d [1 - f n_\alpha^2 - \alpha f (n_\beta^2 - n_\alpha^2)]^{-d/2} \\ &= \rho(f)[\delta_{\alpha\beta} + (1 - \tfrac{1}{4} s(f))(1 - \delta_{\alpha\beta})] \end{aligned} \tag{A.2}$$

with

$$\rho(f) = \tfrac{1}{2}(2\pi)^{-d} K_d^{-1} \int \mathrm{d}\Omega_d (1 - f n_\alpha^2)^{-d/2} = \tfrac{1}{2}(1-f)^{-1/2} \tag{A.3}$$

$$s(f) = 4\left[1 - \frac{2(1-f)^{1/2}}{f} \tan^{-1}\left(\frac{f}{2(1-f)^{1/2}}\right)\right]. \tag{A.4}$$

The anisotropy function $s(f)$ is a monotonic increasing function of f with a large slope for $f \lesssim 1$, $0 \leqslant s(f) < 4$.

Appendix 2

We want to demonstrate that the fixed point equations in the sense of the renormalization group approach can be obtained within the parquet approximation. It is well known from general considerations that in the scaling region the renormalized vertex parts $\Gamma_i(x)$ are proportional to $r^{\epsilon/2} \propto (\epsilon x)^{-1}$ (Wilson and Kogut 1974). Consequently, we make the *ansatz*

$$\Gamma_i(x) = \gamma_i(x)\frac{1}{\epsilon x} \qquad i = 1, 2. \tag{A.5}$$

In the scaling domain the x dependence of the γ_i disappears and the fixed points γ_i^* follow from the condition $\mathrm{d}\gamma_i(x)/\mathrm{d}x = 0$. One gets

$$\begin{aligned} \gamma_1^*\epsilon &= \tfrac{1}{3}(n+8-s)\gamma_1^{*2} + 2\gamma_1^*\gamma_2^* \\ \gamma_2^*\epsilon &= \tfrac{1}{3}s\gamma_1^{*2} + 4\gamma_1^*\gamma_2^* + 3\gamma_2^{*2}. \end{aligned} \tag{A.6}$$

For $s = 0$ the fixed points following from (A.6) coincide with those found by Cowley and Bruce (1973). The fixed points are real. If $s \neq 0$, the fixed points are determined by

$$\begin{aligned} \gamma_1^* &= \frac{\epsilon/2}{2(c+1) \mp (c^2 - s/3)^{1/2}} \\ \gamma_2^* &= \gamma_1^*[c \mp (c^2 - s/3)^{1/2}]. \end{aligned} \tag{A.7}$$

For $\frac{1}{3}s > c^2$ (case (i)) the fixed points become complex.

Appendix 3

The case $n = 1$ can be treated in the simplest way by the addition of the equations (3.7):

$$\tilde{\Gamma}' = (\Gamma_1 + \Gamma_2)' = -3\tilde{\Gamma}^2$$

hence

$$\tilde{\Gamma}(x) = (u+v)[1 + 3(u+v)x]^{-1}. \tag{A.8}$$

Here we use this case as a test of our results of §§ 4 and 5. From (4.3) we find $s_+ = s_- = 3$ and hence case (ii) applies. With $\lambda_1 = -\frac{1}{3}s$, $\theta = -1$ we find

$$\begin{aligned} \Gamma_1(x) &= u\{(v + \tfrac{1}{3}su)[1 + 3(u+v)x]^{-1/3} + (1 - \tfrac{1}{3}s)u\}^{-1}\tilde{\Gamma}(x) \\ \Gamma_2(x) &= u^{-1}\{(v + \tfrac{1}{3}su)[1 + 3(u+v)x]^{-1/3} - \tfrac{1}{3}su\}\Gamma_1(x) \end{aligned} \tag{A.9}$$

and hence $\Gamma_1 + \Gamma_2 = \tilde{\Gamma}$. In fact, the expressions (A.9) show the complicated evolution of the renormalized vertex parts as a function of x (increasing x corresponds to decreasing temperature differences $T - T_c$). In the scaling region we obtain from (A.9)

$$\Gamma_1(x) = \frac{1}{(3-s)x}, \qquad \Gamma_2(x) = -\frac{s}{3(3-s)x} \tag{A.10}$$

in agreement with our result (5.3*b*). The critical exponents are those of the Ising model and do not depend on $s(f)$ as must be the case.

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